Basis functions for general Hsieh-Clough-Tocher triangles,complete or reduced

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SUMMARY

In this note we give the basis function (shape functions) for HSIEH-CLOUGH-TOCHER triangles, complete or reduced. These functions are derived for a general triangle by using the so-called eccentricity parameters.

RESUME

Dans ce travail nous donnons explicitement l'expression des fonctions de base pour les éléments finis de HSIEH-CLOUGH-TOCHER, complets ou réduits. Ces fonctions de base sont obtenues pour un triangle quelconque en utilisant les paramètres d'excentricité.

For submission to Internt. J. Munere.Methods Engng.

1. *I N T R O D U C T I O N.*

The conforming approximation of fourth order problems needs finite elements of class $\mathcal{C}^1$. Focusing our attention to triangular finite elements and, in particular, to those which use polynomial spaces, we have two popular families :

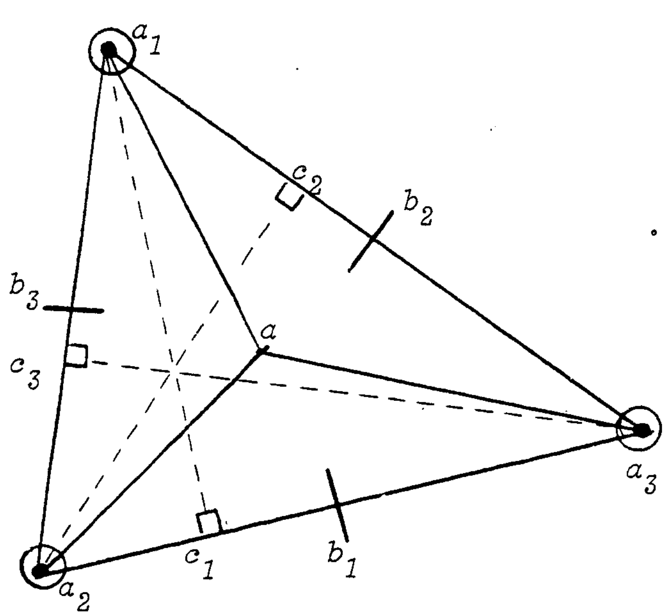
(i) the ARGYRIS triangle [1] which uses complete polynomial of degree five as function space. By suppression of the values of the normal slopes at the three midside nodes, one gets the BELL triangle [2]. The corresponding basis functions of these elements were done by ARGYRIS-FRIED-SCHARPF[1] as TUBA 6 and TUBA 3 elements and next slightly corrected in ARGYRIS-SCHARPF[3]. These Authors achieved considerable simplifications by using the so-called eccentricity parameters which permit to take into account the normal derivatives at the midside nodes (explicitly for ARGYRIS triangle, implicitly for BELL triangle) for triangles of any shape. The high approximation qualities of these elements are well known[4]. Nevertheless, their utilization in the approximation of solutions of fourth order problems requires some regularity for the data and for the solution which is not always satisfied. This fact explains the popularity of the next family of $\mathcal{C}^1$-elements which requires notably less regularity [4] for the data and solution.

(ii) The HSIEH-CLOUGH-TOCHER triangles[5], complete or reduced will be defined in the next paragraph. Their main characteristics are that (1) the triangle is subdivided in three subtriangles using (for example) the center of gravity (2) on each subtriangle we use polynomials of degree three so that the resulting function is of class $\mathcal{C}^1$ on the assembled triangle. The object of this paper is to derive a set of basis functions for both elements, complete or reduced, for triangles of any shape . Similarly to ARGYRIS-FRIED-SCHARPF[1], we use the eccentricity parameters to define the normal slope at midside nodes. These parameters are only dependent on the coordinates of the vertices of the triangle . For simplicity, we shall denote these elements, HCT triangle and reduced HCT triangle, respectively. The implementation of these elements is described in BERNADOU-BOISSERIE-HASSAN [6] and HASSAN [7].

Let us add that (1) a set of basis functions was given by DUPUIS-GOËL [8] for the reduced HCT triangle for triangles of any shape by using a correspondance with a reference triangle, i.e. the unit rectangular triangle. We have checked that our set of basis functions is in complete agreement with their one in the special case of the unit rectangular triangle ; (2) by using the same kind of method it should be possible to derive the basis functions of the FRAEIJS de VEUBEKE-SANDER quadrilateral [9] and of the reduced FRAEIJS de VEUBEKE-SANDER quadrilateral [4].

2. *DEFINITIONS OF HCT AND REDUCED-HCT TRIANGLES.*

We summarize the definitions of these elements in the Figures 1 and 2 using the following notations (see CIARLET [4]) : K, P_K, Σ_K denote respectively the triangle, the space of functions and the set of degrees of freedom ; the indices take values 1,2,3 modulo 3, here, and in the following ; the knowledge of the value of the function, of its first differential, of its normal derivative, in a point is indicated by a black point, a circle surrounding this point, a normal dash, respectively.



$P_K = \left\{ p \in \mathcal{C}^1(K);\ p|_{K_i} \in P_3(K_i),\ i=1,2,3 \right\}$

$\dim P_K = 12$;

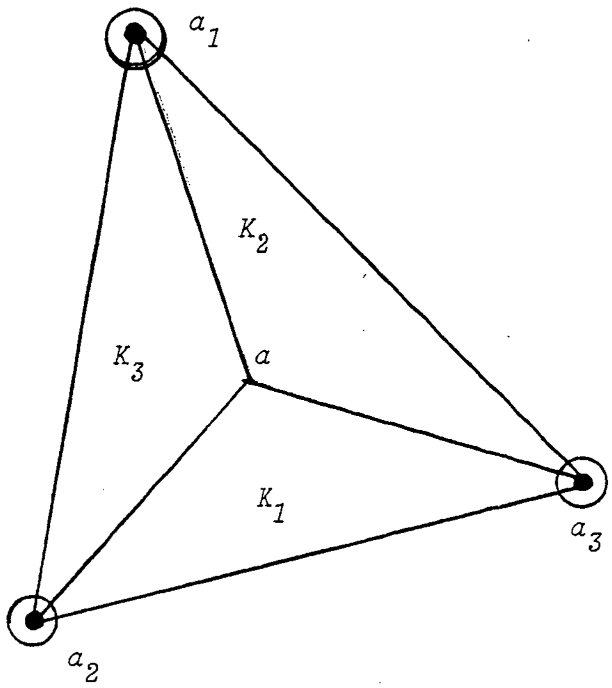
$\Sigma_K = \left\{ p(a_i),\ Dp(a_i)\ (a_{i+1} - a_i),\right.$

$\left. Dp(a_i)\ (a_{i+2} - a_i),\ Dp(b_i)\ (a_i - c_i),\right.$

$\left. i = 1,2,3 \right\}$;

a = barycenter of K.

Figure 1 : HSIEH - CLOUGH - TOCHER triangle



$P_K = \left\{ p \in \mathcal{C}^1(K) \ ; \ p|_{K_i} \in P_3(K_i), \ i = 1,2,3, \right.$

$\left. \frac{\partial p}{\partial n} \in P_1(K')\right.$ for each side K' of $\left. K \right\} ;$

$dim \ P_K = 9 \ ;$

$\Sigma_K = \left\{ p(a_i), \ Dp(a_i) \ (a_{i+1} - a_i), \right.$

$\left. Dp(a_i) \ (a_{i+2} - a_i), \ i = 1,2,3 \right\} ;$

a = barycenter of K.

Figure 2 : REDUCED HSIEH - CLOUGH - TOCHER triangle

3. *SOME NOTATIONS*.

In the general definition of HCT triangles the point a can be any point inside of the triangle. For simplicity, we only consider the case for which the point a is the barycenter of the triangle. With the notations of Figures 1 or 2 we shall denote λ_j (resp. μ_{ij}, $i = 1,2,3$), $j = 1,2,3$, the barycentric coordinates of the triangle $K = \bigcup_{i=1}^{3} K_i$ (resp. K_i, $i = 1,2,3$). We readily check that

$$\mu_{1a} = 3\lambda_1, \ \mu_{12} = \lambda_2 - \lambda_1, \ \mu_{13} = \lambda_3 - \lambda_1 \qquad (1)$$

and two similar relations for the subtriangles K_2 and K_3.

Let c_i (resp. d_i) be the orthogonal projection of the point a_i (resp. a) on the side $a_{i+1} \ a_{i+2}$. Let E_i be the eccentricity parameters of the triangle K ; by definition[1,10]

$$E_i = 2 \, \frac{c_i - b_i}{a_{i+2} - a_{i+1}} = \frac{|\overrightarrow{a_i \, a_{i+1}}|^2 - |\overrightarrow{a_i \, a_{i+2}}|^2}{|\overrightarrow{a_{i+1} \, a_{i+2}}|}, \ i = 1,2,3. \qquad (2)$$

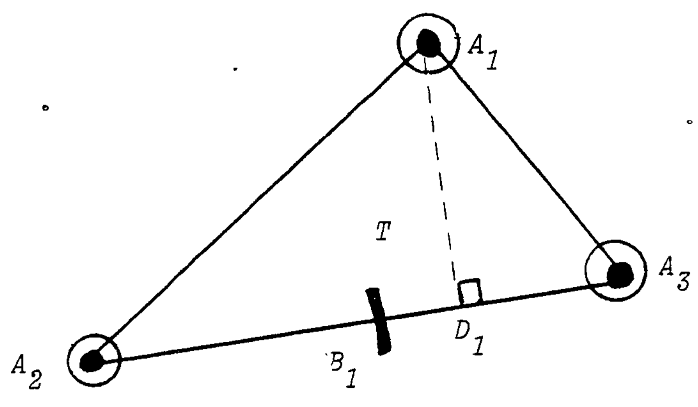
By analogy, we call e_i the eccentricity parameter of the subtriangle K_i, attached to the vertex a, i.e.,

$$e_i = 2\,\frac{d_i - b_i}{a_{i+2} - a_{i+1}}, \quad i = 1,2,3. \tag{3}$$

Let us emphasize that for the subtriangle K_i we just consider the <u>only</u> eccentricity parameter that we need in the following. Evidently we get

$$E_i = 3e_i, \quad i = 1,2,3. \tag{4}$$

In the following it will be convenient to use the finite element described by Figure 3. The triangle T will play successively the role of the subtriangles K_1, K_2, K_3.



$$P_T = P_3(T), \quad ; \qquad \dim P_T = 10\ ;$$

$$\Sigma_T = \Big\{ p(A_i),\ Dp(A_i)\,(A_{i+2} - A_i),\ Dp(A_i)\,(A_{i+1} - A_i)\ ,\ i = 1,2,3,$$
$$D_p(B_1)\,(A_1 - D_1) \Big\}.$$

<u>Figure 3</u> .

Following PEANO [11], a basis for the space P_T is given by the set of the homogeneous monomials

$$\left\{ \eta_1^{i_1} \; \eta_2^{i_2} \; \eta_3^{i_3} \; , \; i_1 + i_2 + i_3 = 3 \right\} \text{ where } \eta_i, \; i = 1,2,3,$$

denote the barycentric coordinates of the triangle T. Then, denoting by $\varpi_T \, v$ the interpolating function of a smooth function v, i.e.,

$$\left. \begin{aligned} \varpi_T \, v = \sum_{i=1}^{3} & \Big\{ v(A_i) \, p_i^0 + Dv(A_i) \; (A_{i+1} - A_i) \; p^1_{i,i+1} \\ & + Dv(A_i) \; (A_{i+2} - A_i) \; p^1_{i,i+2} \Big\} + Dv(B_1) \; (A_1 - D_1) \; p^1_{\perp} \end{aligned} \right\} \tag{5}$$

we see that the basis functions are given by the relation (6).

$$\begin{bmatrix} p_1^0 \\ p_2^0 \\ p_3^0 \\ p^1_{1,2} \\ p^1_{1,3} \\ p^1_{2,3} \\ p^1_{2,1} \\ p^1_{3,1} \\ p^1_{3,2} \\ p^1_{\perp} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 3(1-\alpha_1) \\ & & 1 & 0 & 0 & 0 & 0 & 3 & 3 & 3(1+\alpha_1) \\ & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 & 0 & \frac{3-\alpha_1}{2} \\ & & & & & & 1 & 0 & 0 & -1 \\ & & & & & & & 1 & 0 & -1 \\ & & & & & & & & 1 & \frac{3+\alpha_1}{2} \\ & & & & & & & & & 4 \end{bmatrix} \cdot \begin{bmatrix} \eta_1^3 \\ \eta_2^3 \\ \eta_3^3 \\ \eta_1^2\eta_2 \\ \eta_1^2\eta_3 \\ \eta_2^2\eta_3 \\ \eta_2^2\eta_1 \\ \eta_3^2\eta_1 \\ \eta_3^2\eta_2 \\ \eta_1\eta_2\eta_3 \end{bmatrix} \tag{6}$$

In the relation (6) we denote α_1 the eccentricity parameter attached to the vertex A_1 of the triangle T, i.e.,

$$\alpha_1 = 2\,\frac{D_1 - B_1}{A_3 - A_2}\ . \tag{7}$$

Also, let us emphasize the following result [10,11] which explains the simplicity of the relation (6)

$$Dv(A_i)(A_{i+1} - A_i) = \frac{\partial v}{\partial n_{i+1}}(A_i) - \frac{\partial v}{\partial n_i}(A_i), \quad i = 1,2,3. \tag{8}$$

4. *B A S I S F U N C T I O N S F O R H C T T R I A N G L E.*

Let Π_K be the interpolation operator associated to the HCT triangle and Π_{K_i} the restriction of Π_K to the subtriangle K_i, i.e. $\Pi_{K_i} = \Pi_K|_{K_i}$. By definition $\Pi_{K_i} v \in P_3(K_i)$ for any $v \in \mathcal{C}^1(K)$. Let us set (see figure 1)

$$\left.\begin{aligned}\Pi_{K_i} v = {}& v(a_{i+1})p^0_{i,i+1} + v(a_{i+2})p^0_{i,i+2} + Dv(a_{i+1})(a_{i+2} - a_{i+1})p^1_{i,i+1,i+2}\\ &+ Dv(a_{i+1})(a - a_{i+1})p^1_{i,i+1,a} + Dv(a_{i+2})(a - a_{i+2})p^1_{i,i+2,a}\\ &+ Dv(a_{i+2})(a_{i+1} - a_{i+2})p^1_{i,i+2,i+1} + Dv(b_i)(a - d_i)p^1_{\perp i} + q_i\end{aligned}\right\} \tag{9}$$

where the polynomials $p^0_{i,i+1},\dots,$ are directly derived from (1)(4)(5)(6) and given by (10). The first index i refer to the triangle K_i, the others are directly attached to the definition of the corresponding degrees of freedom.

$$\left.\begin{aligned}
p^0_{i,i+1} &= (\lambda_{i+1}-\lambda_i)\left[(\lambda_{i+1}-\lambda_i)(\lambda_{i+1}+3\lambda_{i+2}+5\lambda_i)+3(3-E_i)\lambda_i(\lambda_{i+2}-\lambda_i)\right]\\
p^0_{i,i+2} &= (\lambda_{i+2}-\lambda_i)\left[(\lambda_{i+2}-\lambda_i)(\lambda_{i+2}+3\lambda_{i+1}+5\lambda_i)+3(3+E_i)\lambda_i(\lambda_{i+1}-\lambda_i)\right]\\
p^1_{i,i+1,i+2} &= (\lambda_{i+1}-\lambda_i)\ (\lambda_{i+2}-\lambda_i)\left(\lambda_{i+1}+\frac{7-E_i}{2}\lambda_i\right)\\
p^1_{i,i+1,a} &= 3\lambda_i(\lambda_{i+1}-\lambda_i)(\lambda_{i+1}-\lambda_{i+2})\\
p^1_{i,i+2,a} &= 3\lambda_i(\lambda_{i+2}-\lambda_i)(\lambda_{i+2}-\lambda_{i+1})\\
p^1_{i,i+2,i+1} &= (\lambda_{i+1}-\lambda_i)(\lambda_{i+2}-\lambda_i)\left(\lambda_{i+2}+\frac{7+E_i}{2}\lambda_i\right)\\
p^1_{\perp i} &= 12\lambda_i(\lambda_{i+1}-\lambda_i)(\lambda_{i+2}-\lambda_i)
\end{aligned}\right\} \tag{10}$$

In order to define $\Pi_{K_i} v$ completely, it remains to determine the functions $q_i \in P_3(K_i)$. Taking into account the definition (10) of the functions p_i and the relation (9), we derive (for example) the following set of equations :

$$q_i(a_j) = o \;,\quad i,\ j = 1,2,3,\quad i \neq j, \tag{11}$$

$$Dq_i(a_j) = o \;,\quad i,\ j = 1,2,3,\quad i \neq j, \tag{12}$$

$$Dq_i(b_i)\ (a - d_i) = o \;,\quad i = 1,2,3, \tag{13}$$

$$q_1(a) = q_2(a) = q_3(a), \tag{14}$$

$$Dq_1(a) = Dq_2(a) = Dq_3(a), \tag{15}$$

$$D\Pi_{K_{i+1}} v \left(\frac{a+a_i}{2}\right) (a_{i+2}-a_{i+1}) = D\Pi_{K_{i+2}} v \left(\frac{a+a_i}{2}\right) (a_{i+2}-a_{i+1}),\ i = 1,2,3. \tag{16}$$

It is easy to check that the only homogeneous polynomials of degree 3 with respect to λ_1, λ_2, λ_3 which satisfy the 21 equations (11) (12) (13) are given by

$$q_i(\lambda_1, \lambda_2, \lambda_3) = \lambda_i^2\ (\beta_{i1}\ \lambda_1 + \beta_{i2}\,\lambda_2 + \beta_{i3}\,\lambda_3).$$

Next, using the 6 equations (14) (15) we prove that the function q_i has the following form (we note $\gamma_i = \beta_{ii}$) :

$$q_i = \lambda_i^2 \left[\gamma_i\ \lambda_i + \frac{1}{3}\ (-2\gamma_i + \gamma_{i+1} - 2\gamma_{i+2})\lambda_{i+1} + \frac{1}{3}\ (-2\gamma_i - 2\gamma_{i+1} + \gamma_{i+2})\lambda_{i+2}\right] \tag{17}$$

Finally, we determine the coefficients γ_1, γ_2, γ_3 by writting that the functions q_i satisfy the last three equations (16) :

$$\left.\begin{aligned}
\gamma_i = &- \frac{1}{2}(E_{i+1} - E_{i+2})\ v(a_i) - \frac{1}{2}\ (27 - 4E_i + E_{i+2})\ v(a_{i+1}) \\
&- \frac{1}{2}(27 + 4E_i - E_{i+1})\ v(a_{i+2}) - \frac{1}{12}\ (1 + E_{i+1})\ Dv(a_i)\ (a_{i+2} - a_i) \\
&- \frac{1}{12}(1 - E_{i+2})Dv(a_i)\ (a_{i+1} - a_i) - \frac{1}{12}\ (7 + E_{i+2})\ Dv(a_{i+1})\ (a_i - a_{i+1}) \\
&- \frac{1}{6}(17 - 2E_i)\ Dv(a_{i+1})\ (a_{i+2} - a_{i+1}) - \frac{1}{6}(17 + 2E_i)\ Dv(a_{i+2})\ (a_{i+1} - a_{i+2}) \\
&- \frac{1}{12}(7 - E_{i+1})\ Dv(a_{i+2})\ (a_i - a_{i+2}) - 8\ Dv(b_i)\ (a - d_i) \\
&- 2\ Dv(b_{i+1})\ (a - d_{i+1}) - 2\ Dv(b_{i+2})\ (a - d_{i+2})
\end{aligned}\right\} \tag{18}$$

Thus, if we set

$$\Pi_{K_i} v = \sum_{j=1}^{3} \Big[v(a_j)\, r^0_{i,j} + Dv(a_j)\,(a_{j+1} - a_j)\, r^1_{i,j,j+1}$$

$$+ Dv(a_j)\,(a_{j+2} - a_j)\, r^1_{i,j,j+2} + Dv(b_j)\,(a_j - c_j)\, r^1_{\perp i,j} \Big] \tag{19}$$

We get the functions r_i by assembling the relations (9) (17) (18). The result is summarized in the tableau 1.

Let us emphasize that (1) these functions r_i are the restrictions of the basis functions of the HCT triangle to the subtriangle K_i ; (2) the knowledge of the functions r_i completely determine Π_{K_i} and hence, completely determine the interpolation operator Π_K.

$$\begin{bmatrix} r^0_{i,i} \\ r^0_{i,i+1} \\ r^0_{i,i+2} \\ r^1_{i,i,i+2} \\ r^1_{i,i,i+1} \\ r^1_{i,i+1,i} \\ r^1_{i,i+1,i+2} \\ r^1_{i,i+2,i+1} \\ r^1_{i,i+2,i} \\ r^1_{\perp i,i} \\ r^1_{\perp i,i+1} \\ r^1_{\perp i,i+2} \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2}(E_{i+1}-E_{i+2}) & 0 & 0 & \frac{3}{2}(3+E_{i+1}) & \frac{3}{2}(3-E_{i+2}) & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}(1-2E_i-E_{i+2}) & 1 & 0 & -\frac{3}{2}(1-E_i) & \frac{3}{2}(E_i+E_{i+2}) & 3 & 3 & 0 & 0 & 3(1-E_i) \\ \frac{1}{2}(1+2E_i+E_{i+1}) & 0 & 1 & -\frac{3}{2}(E_i+E_{i+1}) & -\frac{3}{2}(1+E_i) & 0 & 0 & 3 & 3 & 3(1+E_i) \\ -\frac{1}{12}(1+E_{i+1}) & 0 & 0 & \frac{1}{4}(7+E_{i+1}) & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{12}(1-E_{i+2}) & 0 & 0 & -\frac{1}{2} & \frac{1}{4}(7-E_{i+2}) & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{12}(7+E_{i+2}) & 0 & 0 & \frac{1}{2} & \frac{1}{4}(5+E_{i+2}) & 1 & 0 & 0 & 0 & -1 \\ \frac{1}{6}(4-E_i) & 0 & 0 & -\frac{1}{4}(3-E_i) & -\frac{1}{4}(5-E_i) & 0 & 1 & 0 & 0 & \frac{1}{2}(3-E_i) \\ \frac{1}{6}(4+E_i) & 0 & 0 & -\frac{1}{4}(5+E_i) & -\frac{1}{4}(3+E_i) & 0 & 0 & 1 & 0 & \frac{1}{2}(3+E_i) \\ -\frac{1}{12}(7-E_{i+1}) & 0 & 0 & \frac{1}{4}(5-E_{i+1}) & \frac{1}{2} & 0 & 0 & 0 & 1 & -1 \\ \frac{4}{3} & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 4 \\ -\frac{2}{3} & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{3} & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda_i^3 \\ \lambda_{i+1}^3 \\ \lambda_{i+2}^3 \\ \lambda_i^2\,\lambda_{i+2} \\ \lambda_i^2\,\lambda_{i+1} \\ \lambda_{i+1}^2\,\lambda_i \\ \lambda_{i+1}^2\,\lambda_{i+2} \\ \lambda_{i+2}^2\,\lambda_{i+1} \\ \lambda_{i+2}^2\,\lambda_i \\ \lambda_i\lambda_{i+1}\,\lambda_{i+2} \end{bmatrix}$$

Tableau 1 : Basis functions (19) for the subtriangle K_i of the HCT triangle

[λ_i (resp. E_i), $i = 1,2,3$ are the barycentric coordinates (resp. eccentricity parameters) of the triangle K.]

5. BASIS FUNCTIONS FOR REDUCED HCT-TRIANGLE.

Let $\tilde{\Pi}_K$ be the interpolation operator associated to the reduced-HCT triangle and $\tilde{\Pi}_{K_i}$ the restriction of $\tilde{\Pi}_K$ to the subtriangle K_i, i.e. $\tilde{\Pi}_{K_i} = \tilde{\Pi}_{K|K_i}$. By definition

$$\tilde{\Pi}_{K_i} v \in P_3(K_i), \quad \left.\frac{\partial \tilde{\Pi}_{K_i} v}{\partial n}\right|_{[a_{i+1},\, a_{i+2}]} \in P_1(a_{i+1}, a_{i+2}),$$

for any $v \in \mathcal{C}^1(K)$. Let us set (see figure 2) :

$$\left.\begin{aligned} \tilde{\Pi}_{K_i} v = \sum_{j=1}^{3} \Big[& v(a_j)\, \tilde{r}^0_{i,j} + Dv(a_j)\,(a_{j+1} - a_j)\, \tilde{r}^1_{i,j,j+1} \\ & + Dv(a_j)\,(a_{j+2} - a_j)\, \tilde{r}^1_{i,j,j+2} \Big] \end{aligned}\right\} \tag{20}$$

But $\tilde{\Pi}_{K_i} v \in P_3(K_i)$ implies with the notations of the previous paragraph

$$\Pi_{k_i}(\tilde{\Pi}_{K_i} v) = \tilde{\Pi}_{K_i} v, \qquad \forall\, v \in \mathcal{C}^1(K).$$

Then, the expression (19) involves

$$\left.\begin{aligned} \tilde{\Pi}_{K_i} v = \sum_{j=1}^{3} \Big[& v(a_j)\, r^0_{i,j} + Dv(a_j)\,(a_{j+1} - a_j)\, r^1_{i,j,j+1} \\ & + Dv(a_j)\,(a_{j+2} - a_j)\, r^1_{i,j,j+2} + D\tilde{\Pi}_{K_i} v(b_j)\,(a_j - c_j) r^1_{\perp i,j} \Big] \end{aligned}\right\} \tag{21}$$

By definition $D\tilde{\Pi}_{K_i} v(.)\,(a_j - c_j) \in P_1\,[a_{j+1}, a_{j+2}]$, hence :

$$\begin{aligned} D\tilde{\Pi}_{K_i} v(b_j)\,(a_j - c_j) &= \tfrac{1}{2}\, D\tilde{\Pi}_{K_i} v(a_{j+1})(a_j - c_j) + \tfrac{1}{2}\, D\tilde{\Pi}_{K_i} v(a_{j+2})(a_j - c_j) \\ &= \tfrac{1}{2}\, Dv(a_{j+1})(a_j - c_j) + \tfrac{1}{2}\, Dv(a_{j+2})(a_j - c_j). \end{aligned}$$

Moreover, from (2) we derive

$$a_j - c_j = \frac{1-E_j}{2}(a_j-a_{j+1}) + \frac{1+E_j}{2}(a_j-a_{j+2}),$$

hence

$$\left.\begin{aligned} D\widetilde{\Pi}_{K_i}v(b_j)(a_j-c_j) &= \frac{1-E_j}{4}\Big[Dv(a_{j+1})(a_j-a_{j+1}) + Dv(a_{j+2})(a_j-a_{j+1})\Big] \\ &+ \frac{1+E_j}{4}\Big[Dv(a_{j+1})(a_j-a_{j+2}) + Dv(a_{j+2})(a_j-a_{j+2})\Big] \end{aligned}\right\} \quad (22)$$

Substituting (22) into (21), we get by identification with (20) :

$$\tilde{r}_{i,j} = r^0_{i,j}$$

$$\left.\begin{aligned} \tilde{r}^1_{i,j,j+1} &= r^1_{i,j,j+1} + \frac{1}{2}\, r^1_{\perp i,j+1} - \frac{1+E_{j+2}}{4}\, r^1_{\perp i,j+2} \\ \tilde{r}^1_{i,j,j+2} &= r^1_{i,j,j+2} + \frac{1}{2}\, r^1_{\perp i,j+2} - \frac{1-E_{j+1}}{4}\, r^1_{\perp i,j+1} \end{aligned}\right\} \quad (23)$$

valid for $i,j = 1,2,3.$

Then, using Tableau 1, we derive the functions $\tilde{r}_i$ which are summarized in the Tableau 2.

Here also, we emphasize that (1) <u>these functions</u> $\tilde{r}_i$ <u>are the restrictions of the functions of the reduced HCT-triangle to the subtriangle</u> K_i ; (2) <u>the knowledge of the functions</u> $\tilde{r}_i$ <u>completely determine</u> $\widetilde{\Pi}_{K_i}$ <u>and hence, completely determine the interpolation operator</u> $\widetilde{\Pi}_K$.

$$\begin{bmatrix} \tilde{r}^0_{i,i} \\ \tilde{r}^0_{i,i+1} \\ \tilde{r}^0_{i,i+2} \\ \tilde{r}^1_{i,i,i+2} \\ \tilde{r}^1_{i,i,i+1} \\ \tilde{r}^1_{i,i+1,i} \\ \tilde{r}^1_{i,i+1,i+2} \\ \tilde{r}^1_{i,i+2,i+1} \\ \tilde{r}^1_{i,i+2,i} \\ \\ \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(E_{i+1}-E_{i+2}) & 0 & 0 & \frac{3}{2}(3+E_{i+1}) & \frac{3}{2}(3-E_{i+2}) & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}(1-2E_i-E_{i+2}) & 1 & 0 & -\frac{3}{2}(1-E_i) & \frac{3}{2}(E_i+E_{i+2}) & 3 & 3 & 0 & 0 & 3(1-E_i) \\ \frac{1}{2}(1+2E_i+E_{i+1}) & 0 & 1 & -\frac{3}{2}(E_i+E_{i+1}) & -\frac{3}{2}(1+E_i) & 0 & 0 & 3 & 3 & 3(1+E_i) \\ -\frac{1}{4}(1+E_{i+1}) & 0 & 0 & \frac{1}{4}(5+3E_{i+1}) & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4}(1-E_{i+2}) & 0 & 0 & \frac{1}{2} & \frac{1}{4}(5-3E_{i+2}) & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4}(1-E_{i+2}) & 0 & 0 & -\frac{1}{2} & -\frac{1}{4}(1-3E_{i+2}) & 1 & 0 & 0 & 0 & 1 \\ -\frac{1}{2}E_i & 0 & 0 & -\frac{1}{4}(1-3E_i) & \frac{1}{4}(1+3E_i) & 0 & 1 & 0 & 0 & \frac{1}{2}(1-3E_i) \\ \frac{1}{2}E_i & 0 & 0 & \frac{1}{4}(1-3E_i) & -\frac{1}{4}(1+3E_i) & 0 & 0 & 1 & 0 & \frac{1}{2}(1+3E_i) \\ \frac{1}{4}(1+E_{i+1}) & 0 & 0 & -\frac{1}{4}(1+3E_{i+1}) & -\frac{1}{2} & 0 & 0 & 0 & 1 & 1 \\ \\ \end{bmatrix} \begin{bmatrix} \lambda_i^3 \\ \lambda_{i+1}^3 \\ \lambda_{i+2}^3 \\ \lambda_i^2\lambda_{i+2} \\ \lambda_i^2\lambda_{i+1} \\ \lambda_{i+1}^2\lambda_i \\ \lambda_{i+1}^2\lambda_{i+2} \\ \lambda_{i+2}^2\lambda_{i+1} \\ \lambda_{i+2}^2\lambda_i \\ \lambda_i\lambda_{i+1}\lambda_{i+2} \end{bmatrix}$$

Tableau 2 : Basis functions (20) for the subtriangle K_i of the reduced-HCT triangle

[λ_i (*resp* E_i), $i = 1,2,3$ are the barycentric coordinates (resp. the eccentricity parameters) of the triangle K.]

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REFERENCES

1. J.H. ARGYRIS, I. FRIED and D.W. SCHARPF, *The TUBA family of plate elements for the matrix displacement method, Aeronaut. J. ROY. Aeronaut. Soc., 72 (1968), 701-709.*

2. K. BELL, *A refined triangular plate bending finite element, Internat. J. Numer. Methods Engng., 1 (1969), 101-122.*

3. J.H. ARGYRIS and D.W. SCHARPF, *Some general considerations on the natural mode technique, part 1 : Small displacements, Aeronaut.J. Roy. Aeronaut. Soc., 73 (1969), 218-226.*

4. P.G. CIARLET, *The Finite Element Method for Elliptic Problems, North Holland, Amsterdam, 1978.*

5. R.W. CLOUGH, J.L. TOCHER, *Finite element stiffness matrices for analysis of plates in bending, in Proceedings of the Conference on Matrix Methods in Structural Mechanics, Wright Patterson A.F.B. Ohio, 1965.*

6. M. BERNADOU, J.M. BOISSERIE, K. HASSAN, *Sur l'implémentation des éléments finis de HSIEH-CLOUGH-TOCHER complet et réduit- Rapport de recherches INRIA N°4 (1980)*

7. K. HASSAN, *Les éléments finis du type HSIEH-CLOUGH-TOCHER pour les coques, thèse de doctorat d'Ingénieur, Université Paul Sabatier, Toulouse, France, 1979.*

8. G. DUPUIS, J.J. GOËL, *Finite element with high degree of regularity, Internat. J. Numer. Methods Engng., 2 (1970), 563-577.*

9. B. FRAEIJS de VEUBEKE, *A conforming finite element for plate bending, Internat. J. Solids Structures, 4,(1968), 95-108.*

10. J.H. ARGYRIS, M. HAASE and G.A. MALEJANNAKIS, *Natural geometry of surfaces with specific reference to the matrix displacement analysis of shells, I, II and III, Proc. Kon. Ned. Akad. Wetensch, B76 (1973), 361-410.*

11. A. PEANO, *Hierarchies of conforming finite elements for plane elasticity and plate bending, Comp. Maths. with Appls., 2 (1976), 211-224.*